Summary of Vertical Velocity Calculation Methods as Considered for the 3D ADCIRC Hydrodynamic Model

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OBJECTIVE

To solve the three-dimensional, continuity equation for the vertical velocity given prior solutions for the horizontal velocities and subject to the kinematic boundary conditions at the free surface and the bottom of the water column.

GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

The relevant governing equation is the three-dimensional continuity equation:

$$\frac{\partial u}{\partial x_z}+\frac{\partial v}{\partial y_z}+\frac{\partial w}{\partial z}=0 \qquad (1)$$

where u, v, w define the velocity components in the coordinate directions x, y, z. The subscript "z" has been added to the horizontal derivatives to emphasize that these derivatives are computed in a level coordinate system. We desire to solve this equation for w subject to the free-surface and bottom kinematic boundary conditions:

$$w_s=\frac{\partial \zeta}{\partial t}-u_s\frac{\partial \zeta}{\partial x_z}-v_s\frac{\partial \zeta}{\partial y_z} \quad at \;\; z=\zeta \qquad (2)$$

$$w_b=-u_b\frac{\partial h}{\partial x_z}-v_b\frac{\partial h}{\partial y_z} \quad at \;\; z=-h \qquad (3)$$

where u_s, v_s, w_s are the velocity components at the free surface ($z=\zeta$) and u_b, v_b, w_b are the velocity components at the bottom ($z=-h$) assuming a slip condition is applied at the base of the water column.

ADCIRC utilizes a generalized stretched vertical coordinate system in which the vertical dimension is transformed from z, ranging from $-h$ to ζ, to σ, ranging from b to a, where b and a are arbitrary constants. (Most models assume $b=-1$, $a=0$. ADCIRC assumes $b=-1$, $a=1$. Herein we carry a and b explicitly for the sake of generality.) Using the chain rule we can relate derivatives in the level (z) coordinate reference frame to derivatives in the stretched (σ) coordinate reference frame:

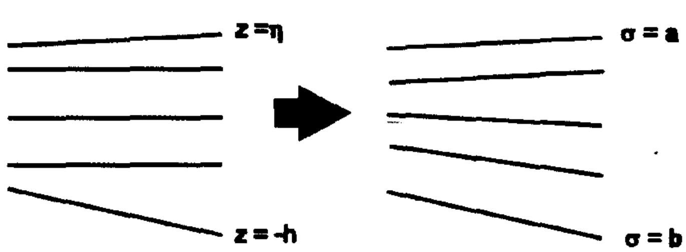

Figure 1. Schematic of level and stretched coordinates

RESULTS

Idealized Inlet problem – horizontal velocity and elevation solutions provided by QUODDY

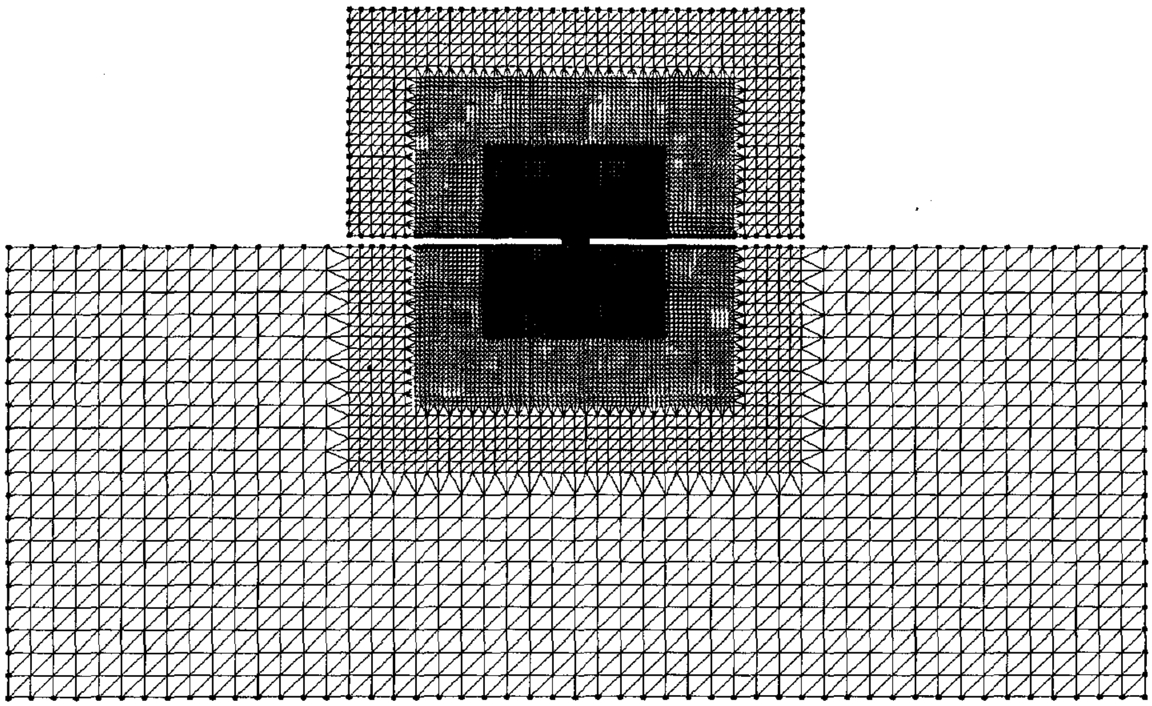

Figure 3. Idealized inlet grid

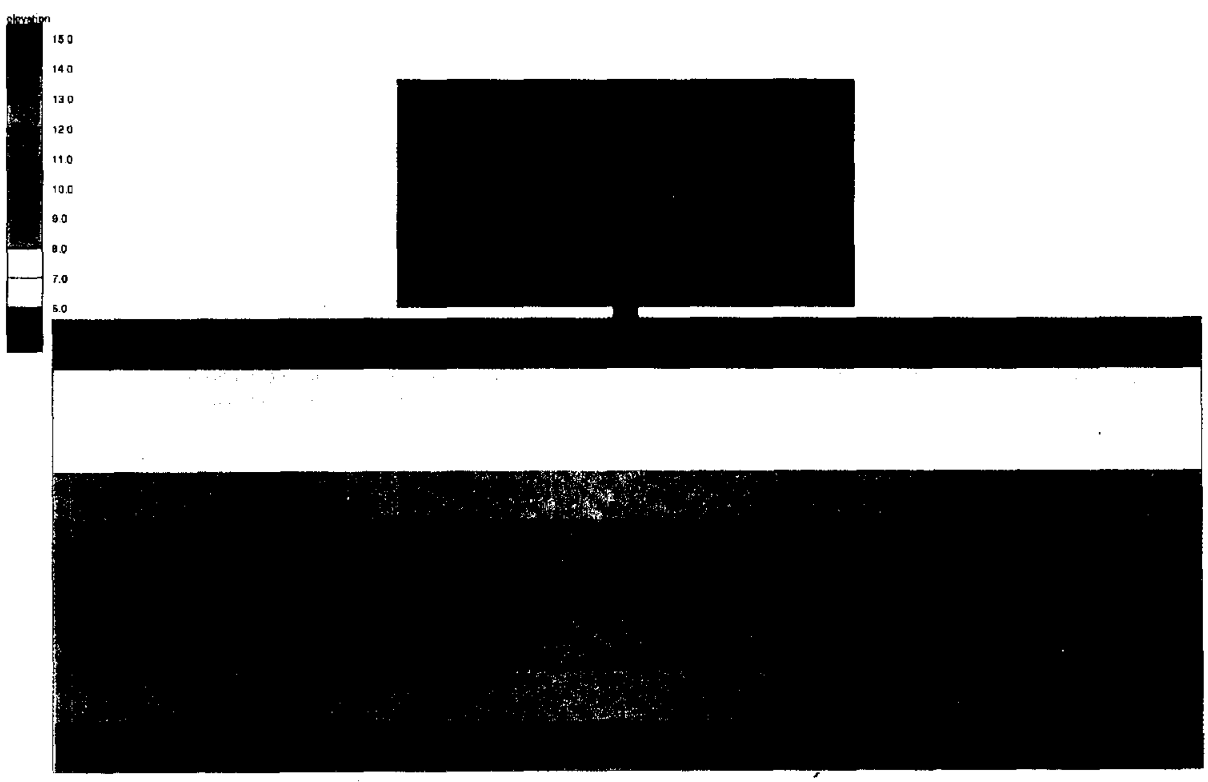

Figure 4. Idealized inlet bathymetry

$$\frac{\partial u}{\partial x_z}=\frac{\partial u}{\partial x_\sigma}-\left[\left(\frac{\sigma-b}{a-b}\right)\frac{\partial\zeta}{\partial x_z}+\left(\frac{\sigma-a}{a-b}\right)\frac{\partial h}{\partial x_z}\right]\frac{\partial u}{\partial z} \tag{4}$$

$$\frac{\partial v}{\partial y_z}=\frac{\partial v}{\partial y_\sigma}-\left[\left(\frac{\sigma-b}{a-b}\right)\frac{\partial\zeta}{\partial y_z}+\left(\frac{\sigma-a}{a-b}\right)\frac{\partial h}{\partial y_z}\right]\frac{\partial v}{\partial z} \tag{5}$$

$$\frac{\partial u}{\partial z}=\frac{a-b}{H}\frac{\partial u}{\partial\sigma}\qquad and\qquad \frac{\partial v}{\partial z}=\frac{a-b}{H}\frac{\partial v}{\partial\sigma} \tag{6}$$

where the total water column depth, $H\equiv h+\zeta$, has been introduced in Eq. (6).

Using Eqs. (2) – (6), the 3D continuity equation in stretched vertical coordinates is:

$$\frac{\partial\zeta}{\partial t}+\frac{\partial(uH)}{\partial x_\sigma}+\frac{\partial(vH)}{\partial y_\sigma}+(a-b)\frac{\partial\omega}{\partial\sigma}=0 \tag{7}$$

while the kinematic free surface and bottom boundary conditions simplify to:

$$\omega=0\quad at\quad \sigma=a \tag{8}$$

$$\omega=0\quad at\quad \sigma=b \tag{9}$$

Eqs. (7) – (9) introduce a stretched-coordinate, vertical velocity, ω, that is related to the true vertical velocity by:

$$\omega\equiv w-\left(\frac{\sigma-b}{a-b}\right)\frac{\partial\zeta}{\partial t}-u\left[\left(\frac{\sigma-b}{a-b}\right)\frac{\partial\zeta}{\partial x_z}+\left(\frac{\sigma-a}{a-b}\right)\frac{\partial h}{\partial x_z}\right]-v\left[\left(\frac{\sigma-b}{a-b}\right)\frac{\partial\zeta}{\partial y_z}+\left(\frac{\sigma-a}{a-b}\right)\frac{\partial h}{\partial y_z}\right] \tag{10}$$

As discussed by Muccino et al. (1997), the solution of Eq. (1) for w with the boundary conditions in Eqs. (2), (3) is an over determined problem, since a first order differential equation admits only one boundary condition constraint on the solution. It is clear that the same problem exists with the solution of Eq. (7) for ω using the boundary conditions in Eqs. (8), (9). Previous modelers have dealt with this problem either by ignoring one of the boundary conditions (e.g., solving Eq. (1) and satisfying only the bottom boundary condition) or by taking a vertical derivative of the 3D continuity, thereby creating a second order differential equation that allows the introduction of both surface and bottom boundary conditions (Lynch and Werner, 1987, 1991). Differentiating over the vertical yields:

$$\frac{\partial^2 w}{\partial z^2}=-\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x_z}+\frac{\partial v}{\partial y_z}\right) \tag{11}$$

$$\frac{\partial^2 \omega}{\partial \sigma^2} = -\frac{1}{(a-b)}\frac{\partial}{\partial \sigma}\left(\frac{\partial \zeta}{\partial t} + \frac{\partial (uH)}{\partial x_\sigma} + \frac{\partial (vH)}{\partial y_\sigma}\right) \tag{12}$$

for Eqs. (1) and (7), respectively. Muccino et al., (1997) suggest two additional method for computing w from Eqs. (1) - (3): a least squares approach and an adjoint approach. Both methods solve the over determined problem in a "best fit" sense with the least squares method operating on the discrete equations and the adjoint method operating on the continuous equations. In test problems, Muccino et al. (1997) found that the least squares and the adjoint approaches yielded essentially identical numerical solutions for w and that this solution was preferable to that obtained by ignoring one boundary condition or computing w using the second order version of the continuity equation.

It is insightful to briefly review the form of the solution that is obtained using the adjoint method:

$$w_{adj} = w_1 + \left(w_s - w_1(\zeta)\right)\frac{L+h+z}{2L+H} \tag{13}$$

where L weights the relative contribution of the boundary conditions vs the interior solution in determining the "best fit". Eq. (13) indicates that the vertical velocity obtained using the adjoint method, w_{adj}, is constructed as the sum of the solution to Eq. (1) satisfying the bottom boundary condition, w_1, and a correction that is proportional to the misfit between w_1 at the free surface and the free surface boundary condition, w_s. Furthermore, Eq. (13) indicates that the correction term varies linearly over the depth. In the limit of $L = 0$, (which places all of the weight on the boundary conditions and eliminates any influence of the interior solution), Eq. (13) reduces to:

$$w_{adj} = w_1 + \left(w_s - w_1(\zeta)\right)\frac{h+z}{H} \qquad for\ L=0 \tag{14}$$

In this case the adjoint correction is a linear function of depth that is zero at the bottom and equal to the surface boundary condition misfit at the free surface. Consequently, the adjoint solution exactly satisfies both the bottom and surface boundary conditions. In the limit of $L \rightarrow \infty$, (which places all of the weight on the interior solution), Eq. (13) reduces to:

$$w_{adj} \rightarrow w_1 + \left(w_s - w_1(\zeta)\right)\frac{1}{2} \qquad for\ L \rightarrow \infty \tag{15}$$

In this case the adjoint correction approaches a constant over the depth that is equal to the average value of the boundary condition misfit at the free surface. Clearly, intermediate values of L generate a correction that falls between these limits.

Note, an adjoint correction is easily derived for the stretched-coordinate vertical velocity:

$$\omega_{adj} = \omega_1 - \omega_1(\zeta)\left[\frac{(\sigma - b) + \dfrac{HL}{a-b}}{(a-b) + \dfrac{2HL}{a-b}}\right] \qquad (16)$$

This correction has the same basic behavior as described above.

NUMERICAL IMPLEMENTATION

The previous section provides several possible approaches for use in determining w and ω in ADCIRC. Testing of various of these options is described below. In each case a vertical sequence of three nodes indicated by superscripts i-1, i, i+1 is assumed. Superscripts - and + indicate quantities evaluated over the intervals {i-1, i} and {i, i+1), respectively, (e.g., $\Delta z^+ \equiv z^{i+1} - z^i$). Since ADCIRC utilizes stretched coordinates, $\Delta z = H\Delta\sigma/(a\text{-}b)$.

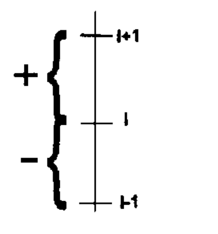

Figure 2. notation used in vertical discretization

(i) Second Derivative approach using Eqs. (11), (2), (3).
Using centered, finite differences, Eq. (11) can be discretized as:

$$\frac{\partial^2 w}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial w}{\partial z}\right) = \frac{\left(\dfrac{\partial w}{\partial z}\right)^+ - \left(\dfrac{\partial w}{\partial z}\right)^-}{\frac{1}{2}(\Delta z^+ + \Delta z^-)} = \frac{\left(\dfrac{w^{i+1} - w^i}{\Delta z^+}\right) - \left(\dfrac{w^i - w^{i-1}}{\Delta z^-}\right)}{\frac{1}{2}(\Delta z^+ + \Delta z^-)} = \qquad (17)$$

$$\frac{2}{(\Delta z^+ + \Delta z^-)}\left[\frac{w^{i+1}}{\Delta z^+} - w^i\left(\frac{1}{\Delta z^+} + \frac{1}{\Delta z^-}\right) + \frac{w^{i-1}}{\Delta z^-}\right]$$

and

$$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x_z} + \frac{\partial v}{\partial y_z}\right) = \frac{\left(\dfrac{\partial u}{\partial x_z} + \dfrac{\partial v}{\partial y_z}\right)^+ - \left(\dfrac{\partial u}{\partial x_z} + \dfrac{\partial v}{\partial y_z}\right)^-}{\frac{1}{2}(\Delta z^+ + \Delta z^-)} =$$

$$\frac{\frac{1}{2}\left[\left(\dfrac{\partial u}{\partial x_z} + \dfrac{\partial v}{\partial y_z}\right)^{i+1} + \left(\dfrac{\partial u}{\partial x_z} + \dfrac{\partial v}{\partial y_z}\right)^{i}\right] - \frac{1}{2}\left[\left(\dfrac{\partial u}{\partial x_z} + \dfrac{\partial v}{\partial y_z}\right)^{i} + \left(\dfrac{\partial u}{\partial x_z} + \dfrac{\partial v}{\partial y_z}\right)^{i-1}\right]}{\frac{1}{2}(\Delta z^+ + \Delta z^-)} \qquad (18)$$

yielding

$$\frac{w^{i+1}}{\Delta z^{+}} - w^{i}\left(\frac{1}{\Delta z^{+}} + \frac{1}{\Delta z^{-}}\right) + \frac{w^{i-1}}{\Delta z^{-}} =$$

$$-\frac{1}{2}\left\{\left[\left(\frac{\partial u}{\partial x_z} + \frac{\partial v}{\partial y_z}\right)^{i+1} + \left(\frac{\partial u}{\partial x_z} + \frac{\partial v}{\partial y_z}\right)^{i}\right]^{+} - \left[\left(\frac{\partial u}{\partial x_z} + \frac{\partial v}{\partial y_z}\right)^{i} + \left(\frac{\partial u}{\partial x_z} + \frac{\partial v}{\partial y_z}\right)^{i-1}\right]^{-}\right\} \quad (19)$$

The left hand side of Eq. (19) is efficiently solved using a standard tri-diagonal matrix solver, however, it is not entirely clear how to evaluate the right hand side of Eq. (19). It appears that the QUODDY and FUNDY models simplify the right hand side by canceling the inner terms, yielding:

$$\frac{w^{i+1}}{\Delta z^{+}} - w^{i}\left(\frac{1}{\Delta z^{+}} + \frac{1}{\Delta z^{-}}\right) + \frac{w^{i-1}}{\Delta z^{-}} = -\frac{1}{2}\left\{\left[\left(\frac{\partial u}{\partial x_z} + \frac{\partial v}{\partial y_z}\right)^{i+1}\right]^{+} - \left[\left(\frac{\partial u}{\partial x_z} + \frac{\partial v}{\partial y_z}\right)^{i-1}\right]^{-}\right\} \quad (20)$$

Both Eqs. (19) and (20) require the evaluation of level coordinate horizontal derivatives of horizontal velocity. These are converted to horizontal derivatives in stretched coordinates using Eqs. (4) and (5), thereby requiring the additional evaluation of vertical derivatives of the horizontal velocity. These vertical derivatives are computed as a simple difference over the intervals {i-1, i} or {i, i+1} for evaluation of the terms in $[]^-$ or $[]^+$, respectively. For example,

$$\left[\left(\frac{\partial u}{\partial x_z}\right)^{i+1}\right]^{+} = \left(\frac{\partial u}{\partial x_\sigma}\right)^{i+1} - \left[\left(\frac{\sigma^{i+1} - b}{a-b}\right)\frac{\partial \zeta}{\partial x_z} + \left(\frac{\sigma^{i+1} - a}{a-b}\right)\frac{\partial h}{\partial x_z}\right]\left(\frac{\partial u}{\partial z}\right)^{+} \quad (21)$$

$$\left[\left(\frac{\partial u}{\partial x_z}\right)^{i}\right]^{+} = \left(\frac{\partial u}{\partial x_\sigma}\right)^{i} - \left[\left(\frac{\sigma^{i} - b}{a-b}\right)\frac{\partial \zeta}{\partial x_z} + \left(\frac{\sigma^{i} - a}{a-b}\right)\frac{\partial h}{\partial x_z}\right]\left(\frac{\partial u}{\partial z}\right)^{+} \quad (22)$$

$$\left[\left(\frac{\partial u}{\partial x_z}\right)^{i}\right]^{-} = \left(\frac{\partial u}{\partial x_\sigma}\right)^{i} - \left[\left(\frac{\sigma^{i} - b}{a-b}\right)\frac{\partial \zeta}{\partial x_z} + \left(\frac{\sigma^{i} - a}{a-b}\right)\frac{\partial h}{\partial x_z}\right]\left(\frac{\partial u}{\partial z}\right)^{-} \quad (23)$$

$$\left[\left(\frac{\partial u}{\partial x_z}\right)^{i-1}\right]^{-} = \left(\frac{\partial u}{\partial x_\sigma}\right)^{i-1} - \left[\left(\frac{\sigma^{i-1} - b}{a-b}\right)\frac{\partial \zeta}{\partial x_z} + \left(\frac{\sigma^{i-1} - a}{a-b}\right)\frac{\partial h}{\partial x_z}\right]\left(\frac{\partial u}{\partial z}\right)^{-} \quad (24)$$

Notice that if Eq. (19) is discretized, the inner terms cancel only if $\left(\frac{\partial u}{\partial z}\right)^{+} = \left(\frac{\partial u}{\partial z}\right)^{-}$. Thus, the fully discretized version of Eqs. (19) and (20) are different.

Given a solution for w, • is obtained directly from Eq. (10). The top and bottom rows of the matrix problem generated from Eq. (19) are modified to insert the free surface and bottom boundary conditions into the problem.

(ii) Second Derivative approach using Eqs. (12), (8), (9).
This approach is similar to approach *(i)* except that Eq. (12) is used to produce a solution for • and w then is computed from Eq. (10). This approach is attractive since Eq. (12) is written in terms of horizontal derivatives in the stretched coordinate system, thereby minimizing the additional computations required to convert horizontal derivatives back to level coordinates. However, initial experience with this approach was highly unsatisfactory, presumably because the vertical derivative applied to the terms on the right hand side of Eq. (12) eliminates the time derivative term found there (at least in the initial implementation). This clearly removes an important physical effect from • . It is likely that this problem can be corrected by carefully computing this combined derivative. This could be looked into further.

(iii) First Derivative approach using Eqs. (1), (2), (3), with or without the adjoint correction.
Eq. (1) is discretized as:

$$\frac{\partial w}{\partial z} = \frac{w^i - w^{i-1}}{\Delta z^-} \tag{25}$$

and

$$\frac{\partial u}{\partial x_z} + \frac{\partial v}{\partial y_z} = \frac{1}{2}\left[\left(\frac{\partial u}{\partial x_z} + \frac{\partial v}{\partial y_z}\right)^i + \left(\frac{\partial u}{\partial x_z} + \frac{\partial v}{\partial y_z}\right)^{i-1}\right] \tag{26}$$

yielding

$$\frac{w^i - w^{i-1}}{\Delta z^-} = -\frac{1}{2}\left[\left(\frac{\partial u}{\partial x_z} + \frac{\partial v}{\partial y_z}\right)^i + \left(\frac{\partial u}{\partial x_z} + \frac{\partial v}{\partial y_z}\right)^{i-1}\right] \tag{27}$$

Eq. (26) is solved explicitly for w, using the bottom boundary condition to initiate the calculation. The right hand side of Eq. (26) requires evaluation of level coordinate horizontal derivatives of horizontal velocity. These are converted to horizontal derivatives in stretched coordinates using Eqs. (4), (5), thereby requiring the additional evaluation of vertical derivatives of the horizontal velocity. These vertical derivatives are computed as a simple difference over the interval {i-1, i}. Given a solution for w, • is obtained directly from Eq. (10). If desired the adjoint correction, Eq. (13), can be applied once w has been computed from Eq. (27).

(iv) First Derivative approach using Eqs. (7), (8), (9), with or without the adjoint correction.
This approach is similar to approach *(iii)* except that Eq. (7) is used to produce a solution for • and w then is computed from Eq. (10). This approach is attractive since Eq. (7) is written in terms of horizontal derivatives in the stretched coordinate system, thereby minimizing the

additional computations required to convert horizontal derivatives back to level coordinates. If desired the adjoint correction, Eq. (16), can be applied once • has been computed.

HORIZONTAL DISCRETIZATION

ADCIRC utilizes two different integration rules in implementing the FE method in the horizontal dimension. (In some cases the integration my be further approximated using lumping.) The GWCE equation uses an exact integration, i.e.,

$$\left\langle \gamma,\phi_i \right\rangle_\Omega \equiv \sum_{n=1}^{NE_i} \int_{\Omega_n} \gamma\phi_i \, d\Omega \tag{28}$$

while the momentum equations use an approximate integration

$$\left\langle \gamma,\phi_i \right\rangle_\Omega \cong \frac{A_i}{3NE_i} \sum_{n=1}^{NE_i} \frac{3}{A_n} \int_{\Omega_n} \gamma\phi_i \, d\Omega \tag{29}$$

In these expressions the integration has been applied over each element • $_n$ (n = 1...NE_i) containing node i, A_n is the area of element n and A_i is the total area of all elements containing node i.

If • represents a horizontal derivative (and is therefore constant over an element) and $\int_{\Omega_n} \phi_i \, d\Omega = \frac{A_n}{3}$, Eqs. (28) and (29) simplify to:

$$\left\langle \gamma,\phi_i \right\rangle_\Omega \equiv \frac{1}{3} \sum_{n=1}^{NE_i} A_n \gamma_n \tag{30}$$

and

$$\left\langle \gamma,\phi_i \right\rangle_\Omega \cong \frac{A_i}{3NE_i} \sum_{n=1}^{NE_i} \gamma_n \tag{31}$$

Physically, the difference between the integrations in Eqs. (30) and (31) is that the exact integration computes the integrated horizontal derivative at node i as the sum of the horizontal derivatives in all elements surrounding node i weighted by the particular element's area. The approximate integration computes the integrated horizontal derivative at node i as an unweighted sum of the horizontal derivatives in the elements surrounding node i multiplied by the total area of all elements surrounding node i. The two methods are equal for a uniform grid.

- 2nd derivative options either based on Eq. (19) or Eq. (20) gave seemingly good, smooth and comparable results.
- 1st derivative option gave most reasonable results when the adjoint correction was applied with L=0. In this case the result matched the 2nd derivative option based on Eq. (19) to 5 – 6 decimal places.
- Solutions for *w* are identical to 4 – 5 decimal places (single precision calculations) whether *w* is solved directly or via • .

Vancouver Island problem – horizontal velocity solutions provided by FUNDY

- Results quite sensitive to horizontal integration rule!
- If horizontal derivatives are computed using the exact integration rule, solutions for *w* are identical to 4 – 5 decimal places (single precision calculations) whether *w* is solved directly or via • .
- 2nd derivative option based on Eq. (19) again matches 1st derivative method with L=0.
- 2nd derivative option based on Eq. (20) gives significantly different results, generally stronger upwelling, than other methods as shown by Julia earlier. This did not show up in idealized inlet problem because of very small bathymetric and surface elevation gradients.
- If exact horizontal integration is used, the methodological choices seem to collapse to a choice between the adjoint method (with L=0 or not) and the 2nd derivative method based on Eq. (20).

Quarter Annular Tidal Problem – horizontal velocity and elevation solutions from analytical solution

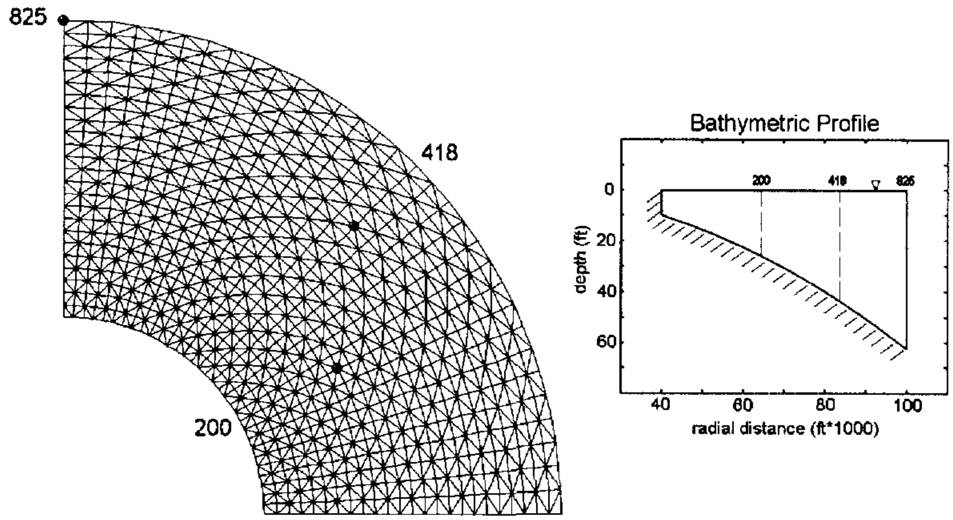

Figure 5. Finite element grid used for the quarter-annular tidal problem

- Outer boundary forced with M2 tide.
- 2nd derivative option based on Eq. (19) again matches 1st derivative method with L=0.
- 1st derivative method with L=0 gives consistently better results than 2nd derivative option using Eq. (20) (QUODDY/FUNDY solution)

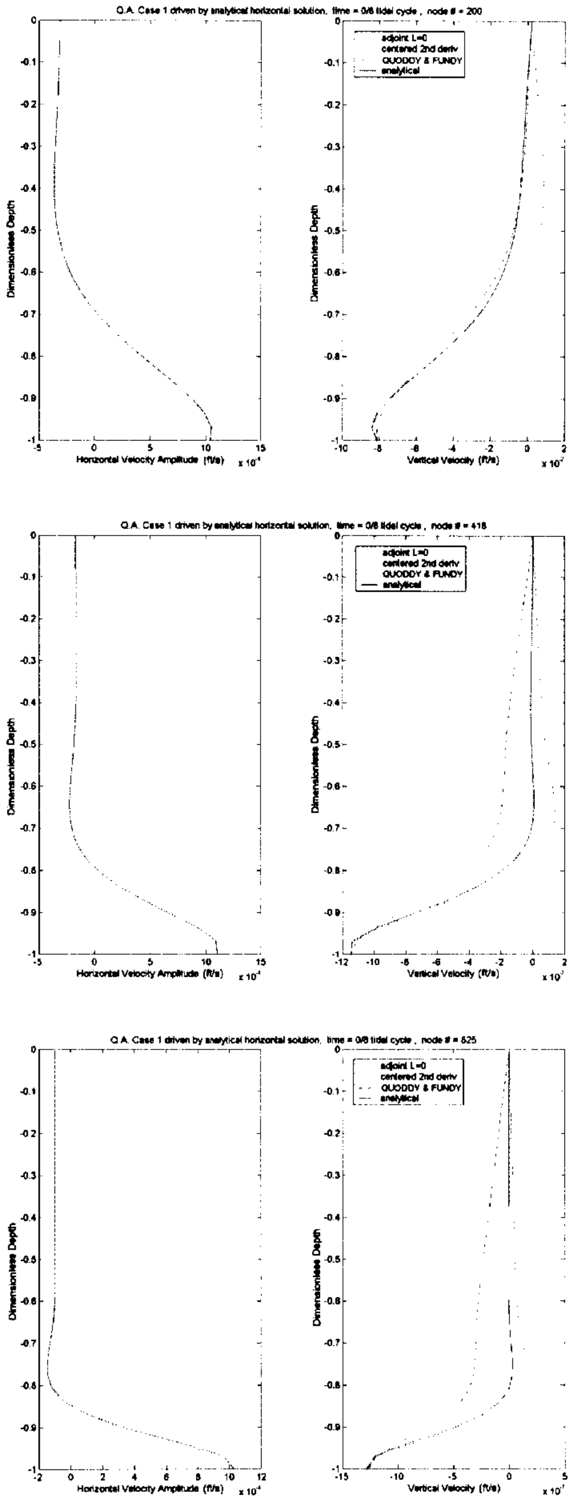

Figure 6. Horizontal and vertical velocities at nodes 200, 418 and 825 at the beginning of the tidal cycle

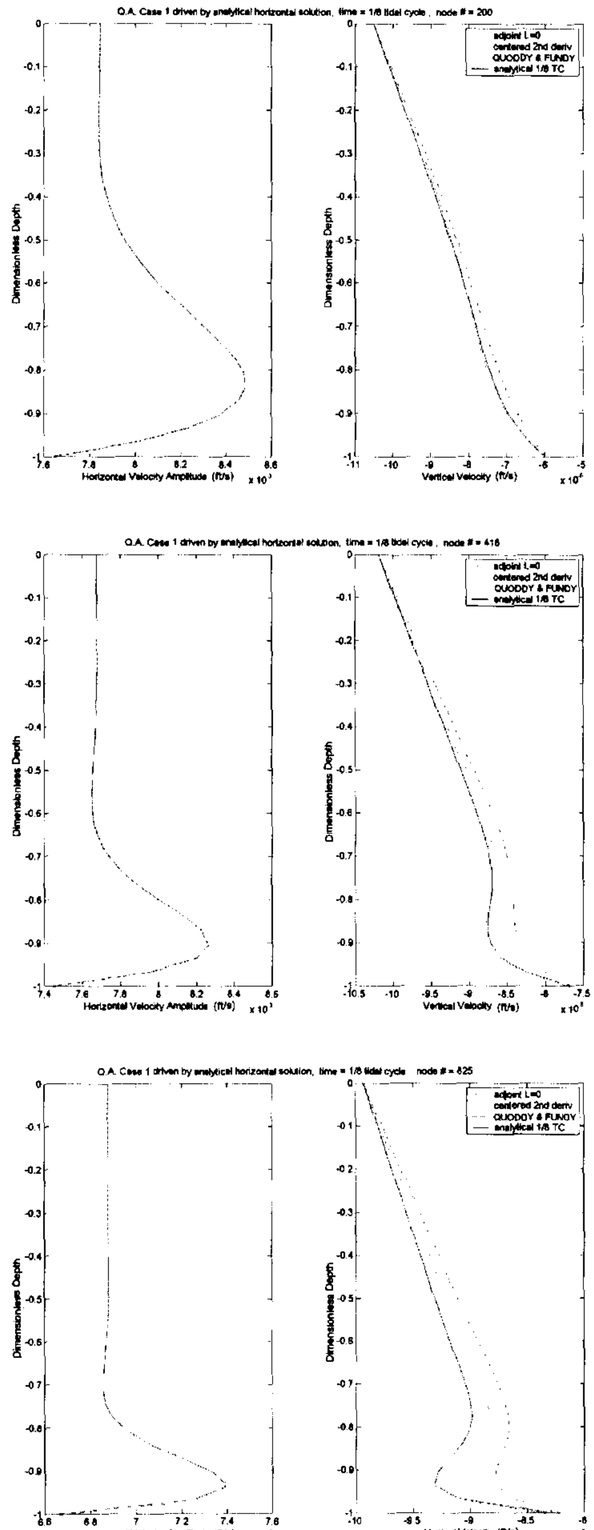

Figure 7. Horizontal and vertical velocities at nodes 200, 418 and 825 at 1/8 of the tidal cycle

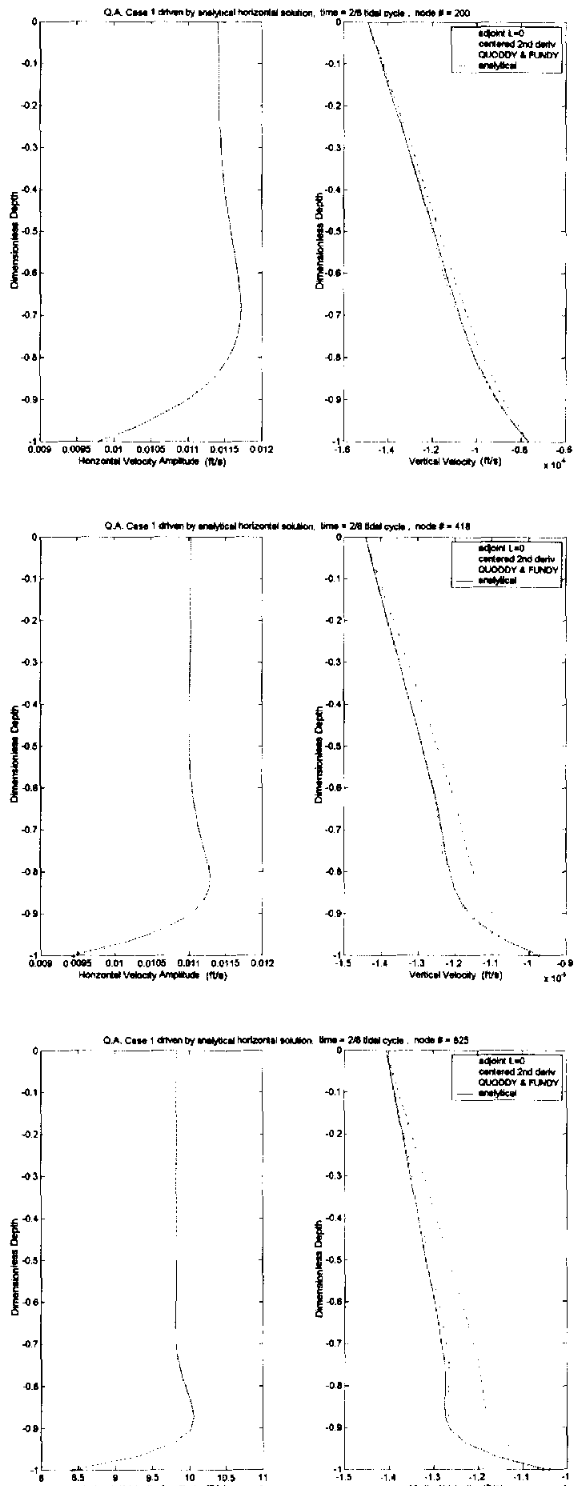

Figure 8. Horizontal and vertical velocities at nodes 200, 418 and 825 at 1/4 of the tidal cycle

DISCUSSION

It appears that out of the multiple methods for computing w, as long as the exact integration is used for the horizontal derivative terms (i) it makes no difference whether one computes ω or w first, and (ii) the 2nd derivative scheme based on Eq. (19) is equivalent to the 1st derivative adjoint scheme with L=0. The first of these findings should be the case and is encouraging to verify. The second finding also seems reasonable, since a 2nd derivative equation should admit an additional term that is linear in depth into the solution beyond that which satisfies the 1st derivative equation. Since the solution of the 2nd derivative equation is forced to satisfy both the surface and bottom boundary conditions, this should be equivalent to adding a linear correction to the 1st derivative equation that is zero at the bottom and causes the solution to exactly satisfy the surface boundary condition. A comparison of the discrete equations further confirms this. If Eq. (27) is subtracted from the equivalent equation for the interval {i, i+1}, the result becomes:

$$\frac{w^{i+1}}{\Delta z^+} - w^i\left(\frac{1}{\Delta z^+}+\frac{1}{\Delta z^-}\right)+\frac{w^{i-1}}{\Delta z^-} = -\frac{1}{2}\left\{\left[\left(\frac{\partial u}{\partial x_z}+\frac{\partial v}{\partial y_z}\right)^{i+1}+\left(\frac{\partial u}{\partial x_z}+\frac{\partial v}{\partial y_z}\right)^{i}\right]^+ - \left[\left(\frac{\partial u}{\partial x_z}+\frac{\partial v}{\partial y_z}\right)^{i}+\left(\frac{\partial u}{\partial x_z}+\frac{\partial v}{\partial y_z}\right)^{i-1}\right]^-\right\} \tag{32}$$

This is identical to the discrete equations for the 2nd derivative approach obtained in Eq. (19). Thus a linear combination of 1st derivative equations is equivalent to the 2nd derivatives, suggesting the solution using either approach will be identical, provided both boundary conditions are enforced.

Therefore, the primary issues appears to be whether to use an adjoint 1st derivative or 2nd derivative (Eq. 19) solution or a 2nd derivative (Eq. 20) approach. Based on comparisons to the analytical solution for w in the quarter annular tidal problem, the adjoint, 1st derivative approach seems preferable.

Vertical velocity calculation is very sensitive to both horizontal discretization and vertical discretization.